

REGULAR GLEASON MEASURES AND GENERALIZED EFFECT ALGEBRAS

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ABSTRACT. We study measures, finitely additive measures, regular measures, and σ -additive measures that can attain even infinite values on the quantum logic of a Hilbert space. We show when particular classes of non-negative measures can be studied in the frame of generalized effect algebras.

1. INTRODUCTION

A basic theoretical tool of quantum mechanical measurements is based on the famous Gleason Theorem [Gle] which says that if H is a separable Hilbert space over real, complex or quaternion numbers, $\dim H \neq 2$, then there is a one-to-one correspondence between σ -additive states, m , on the system $\mathcal{L}(H)$ of all closed subspaces of H , and the set of von Neumann operators on H , i.e. positive Hermitian operators, T , of trace equal to 1, given by

$$m(M) = \text{tr}(TP_M), \quad M \in \mathcal{L}(H), \quad (1.1)$$

where P_M is an orthogonal projector onto M . Formula (1.1) holds also for any completely additive state for every $\mathcal{L}(H)$, $\dim H \neq 2$. For an extension of the Gleason Theorem to non-separable Hilbert spaces, see e.g. [Dvu2, Dvu3]. In such a cases, if $\dim H \neq 2$ is a non-measurable cardinal, then every σ -additive state is completely additive. In addition, formula (1.1) is valid also for bounded signed σ -additive measures, [Dvu3].

However, there are also important measures attaining even infinite values, like, $\dim M$, $M \in \mathcal{L}(H)$, or attaining both positive and negative values, e.g. $\text{tr}(T(AP_M + P_MA)/2)$, $M \in \mathcal{L}(H)$, where A is a Hermitian operator on H that is neither positive nor negative. Also for them it is possible to find an extension of the Gleason formula

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(1.1), however, in such a case, they are induced rather by bilinear forms than by trace operators, see e.g. [LuSh] as it will be shown by formula (3.1). Therefore, we call Gleason measures all measures that can be expressed by a generalized Gleason formula.

If we study finitely measures, m , with improper values, then regularity of m , i.e. the property “given $M \in \mathcal{L}(H)$, the value $m(M)$ can be approximated by values of $m(N)$, where N is a finite-dimensional subspace of M ”, does not imply that the measure is σ -additive as in the case that m is a finite measure. By [Dvu3, Lem 3.7.8. Thm 3.7.9], every positive regular measure m that is σ -finite (i.e. H can be split into a sequence of mutually orthogonal subspaces of finite measure) is in a one-to-one correspondence with densely defined positive bilinear forms. Consequently, regular measures are Gleason measures, too. Since, not every bilinear form is induced by a Hermitian operator, bilinear forms define more regular measures than do bilinear forms induced by Hermitian trace operators via (1.1). For more details on measures on $\mathcal{L}(H)$, we recommend to consult the book [Dvu3].

Foulis and Bennett [FoBe] have introduced effect algebras that are inspired by mathematical models of quantum mechanics. They model both a two-valued reasoning and a many valued one. They have a least and a top element, 0 and 1, respectively, and a basic operation is a partially defined addition $+$. For example, POV-measures have within these models a natural form as observables. More general structures than effect algebras are generalized effect algebras [DvPu], which have a similar partially defined addition, but the top element 1 is not guaranteed. For additional information on effect algebras and generalized effect algebras, we recommend the monograph [Dvu3].

The aim of this paper is to study some properties of measures using methods of generalized effect algebras. We describe a structure of generalized effect algebras of finitely additive Gleason measures, regular measures, and σ -measures. In addition, we study also questions connected with a Dedekind monotone σ -convergence.

The paper is organized as follows. Section 2 presents elements of bilinear forms on a Hilbert space and frame functions that are semibounded. Regular measures and their properties are studied in Section 3 together with different kinds of generalized effect algebras. In Section 4, we deal with monotone convergence properties of σ -additive measures.

2. BILINEAR FORMS, SIGNED MEASURES, AND FRAME FUNCTIONS

Let S be a real or complex inner product space with an inner product (\cdot, \cdot) which is linear in the left argument and antilinear in the right one. Sometimes we call S a pre-Hilbert space. If S is complete with respect to the norm $\|\cdot\|$ induced by the inner product, S is said to be a *Hilbert space*. Let H be a Hilbert space and let $\mathcal{L}(H)$ be the system of all closed subspaces of H . We denote by $\mathbf{0}$ the zero vector of H and if $x_1, \dots, x_n \in H$, $\text{sp}(x_1, \dots, x_n)$ denotes the span generated by x_1, \dots, x_n . We denote by $\mathcal{S}(H)$ the unit sphere of H . Similarly, if S is a linear submanifold of H , $\mathcal{S}(S)$ denotes the unit sphere of S .

Any system of mutually orthogonal unit vectors $\{x_i\}$ of an inner product space S is denoted as ONS. An ONS $\{x_i\}$ is said to be (i) a *maximal* ONS (MONS for short) if x is a vector orthogonal to each vector x_i , then $x = \mathbf{0}$; (ii) *orthonormal base* (ONB for short), if for any vector $x \in S$, we have $x = \sum_i (x, x_i)x_i$.

A Hermitian operator T on H is said to be a *trace operator* (or of *finite trace*) if there is a real constant $\text{tr}(T)$, called the *trace* of T , such that for any orthonormal basis $\{x_i\}$, $\text{tr}(T) = \sum_i (Tx_i, x_i)$. We denote by $\text{Tr}(H)$ the class of Hermitian operators with finite trace.

Now we introduce bilinear forms. Let $D(t)$ be a submanifold of a complex Hilbert space H and let $D(t)$ be dense in H . A mapping $t : D(t) \times D(t) \rightarrow \mathbb{C}$ is called a *bilinear form* (or a *sesquilinear form*) if and only if it is additive in both arguments and $(\alpha x, \beta y) = \alpha \bar{\beta} (x, y)$ for all $\alpha, \beta \in \mathbb{C}$, $x, y \in D(t)$, where $\bar{\beta}$ is the complex conjugation of β . A complex-valued function \hat{t} with the definition domain $D(t)$ is defined by $\hat{t}(x) = t(x, x)$, $x \in D(t)$, and it is said to be a *quadratic form* induced by the bilinear form t , [Hal, p. 12]. A bilinear form t is called *symmetric* if $t(x, y) = \overline{t(y, x)}$.

A symmetric bilinear form t is (i) *semibounded* if $\inf\{t(x, x) : x \in \mathcal{S}(D(t))\} > -\infty$, (ii) *positive* if $t(x, x) \geq 0$, $x \in \mathcal{S}(D(t))$, and (iii) *bounded* if $\sup\{|t(x, x)| : x \in \mathcal{S}(D(t))\} < \infty$; otherwise t is called *unbounded*. We denote by $\mathcal{BF}(H)$ and $\mathcal{PBF}(H)$ the set of all bilinear forms and positive bilinear forms, respectively, on H . We denote by o the bilinear with $D(o) = H$ defined by $o(x, y) = 0$ for all $x, y \in H$; then $o \in \mathcal{PBF}(H)$.

On $\mathcal{BF}(H)$, we can define a *usual sum* $t+s$ for each t, s on $D(t+s) := D(t) \cap D(s)$ by $(t+s)(x, y) := t(x, y) + s(x, y)$ for all $x, y \in D(t) \cap D(s)$, and the multiplication by a scalar $\alpha \in \mathbb{C}$ by $(\alpha t)(x, y) := \alpha t(x, y)$ for $x, y \in D(\alpha t) := D(t)$.

Given two densely defined positive bilinear forms t and s , we write

$$t \preceq s \tag{2.0}$$

if and only if $D(t) \supseteq D(s)$ and $t(x, x) \leq s(x, x)$ for all $x \in D(s)$. Then \preceq is a partial order on $\mathcal{PBF}(\mathcal{H})$, and the bilinear form o is the least element of the set $\mathcal{PBF}(\mathcal{H})$, i.e. $o \preceq t$ for any $t \in \mathcal{PBF}(\mathcal{H})$.

A bilinear form s is an *extension* of a bilinear form t iff $D(t) \subseteq D(s)$ and, for every $x, y \in D(t)$, $t(x, y) = s(x, y)$; we will write $s|_{D(t)} = t$. A bilinear form t is *closable* if it has some closed extension. Then the least closed extension of a closable t is said to be a *closure* and we denote it by \bar{t} .

For any densely defined operator A with domain $D(A)$ on H , the mapping $t(x, y) := (Ax, y)$ is a bilinear form on $D(A)$. We say that a bilinear form t *corresponds* to an operator A iff $D(A) \subseteq D(t)$ and, for every $x, y \in D(A)$, $t(x, y) = (Ax, y)$. We say that t is *generated* by A if t corresponds to A and $D(A) = D(t)$. We note that there are bilinear forms that are not generated by any operator. In addition, there is a one-to-one correspondence between closed positive symmetric bilinear forms, t , and positive self-adjoint operators, A , given by $D(t) = D(A^{1/2})$, and $t(x, y) = (A^{1/2}x, A^{1/2}y)$, $x, y \in D(t)$.

Given a symmetric semibounded bilinear form t , we can equip its domain $D(t)$ with an inner product $(x, y)_t := t(x, y) + (1 + m_t)(x, y)$. In this way $D(t)$ becomes a pre-Hilbert space. Whenever $D(t)$ with $(x, y)_t$ is a Hilbert space, we call t *closed*.

A bilinear form s is an *extension* of a bilinear form t iff $D(t) \subseteq D(s)$ and, for every $x, y \in D(t)$, $t(x, y) = s(x, y)$; we will write $s|_{D(t)} = t$. A bilinear form t is *closable* if it has some closed extension.

An important result which is also a matter of interest for us was proved in [Sim, Thm 2.1, Thm 2.2]:

Theorem 2.1. *Let t be a densely defined positive symmetric bilinear form on a Hilbert space H . Then there exist two positive symmetric bilinear forms t_r and t_s such that $D(t) = D(t_r) = D(t_s)$ such that*

$$t = t_r + t_s,$$

where t_r is the largest closable bilinear form less than t in the ordering \preceq .

In view of Theorem 2.1, the components t_r and t_s of the positive symmetric bilinear form t , are said to be the *regular* and *singular* part of t , respectively.

If t_s is identical zero, the bilinear form t is said to be *regular*, and if t_r is identical zero, t is said to be *singular*. Hence, the bilinear form o is the least positive symmetric bilinear form and is a unique positive symmetric bilinear form that is simultaneously regular and singular.

The following example presents a nonzero everywhere defined positive singular symmetric bilinear form.

Example 2.2. *Let $\{e_n\}$ be an orthonormal basis of an infinite-dimensional Hilbert space H which is a part of a Hamel basis $\{e_i\}_{i \in I}$ of H consisting of unit vectors from H . Fix an element e_{i_0} , $i_0 \in I$, which does not belong to the orthonormal basis $\{e_n\}$, and define a linear operator T on H by*

$$T\left(\sum_i \alpha_i e_i\right) = \alpha_{i_0} e_{i_0},$$

where α_{i_0} is the scalar corresponding to e_{i_0} in the decomposition $x = \sum_i \alpha_i e_i$, $x \in H$, with respect to the given Hamel basis. Then T is an everywhere defined unbounded linear operator.

If we define a bilinear form t with $D(t) = H$ via

$$t(x, y) = (Tx, Ty), \quad x, y \in H,$$

then t is a nonzero everywhere defined positive singular symmetric bilinear form.

A mapping $m : \mathcal{L}(H) \rightarrow [-\infty, \infty]$ is said to be a *finitely additive signed measure* if (i) $m(\text{sp}(\mathbf{0})) = 0$, and (ii) $m(M \vee N) = m(M) + m(N)$ whenever $M \perp N$, $M, N \in \mathcal{L}(H)$. We note that m can attain from the improper values $\{-\infty, \infty\}$ at most one value.

If for $m : \mathcal{L}(H) \rightarrow [-\infty, \infty]$ with the above (i) and instead of (ii) we have (ii)', $m(\bigvee_{i \in I} M_i) = \sum_{i \in I} m(M_i)$ whenever $M_i \perp M_j$ for $i \neq j$, $i, j \in I$ holds for any index set I , m is said to be a *completely additive signed measure* (also totally additive); if it holds only for any countable index set I , m is said to be a *σ -additive signed measure*. If $m(M) \geq 0$ for any $M \in \mathcal{L}(H)$, we are speaking on a *finitely additive measure*, a *completely additive measure*, and a *σ -additive measure*, respectively. Every finitely additive measure is monotone. If, in addition, $m(M) \geq 0$ for any $M \in \mathcal{L}(H)$ and $m(H) = 1$, m is said to be a *finitely additive state*, a *completely additive measure* and a *σ -additive state*, respectively. As we have already mentioned in Introduction, if m is a completely additive state on $\mathcal{L}(H)$, $\dim H \neq 2$, then by the Gleason Theorem, see e.g. [Dvu3, Thm 3.2.21], there is a unique positive Hermitian operator T on H with trace $\text{tr}(T) = 1$ such that

$$m(M) = \text{tr}(TP_M), \quad M \in \mathcal{L}(H). \quad (2.1)$$

If H is a finite-dimensional Hilbert space with $\dim H \neq 2$ and m is a finite finitely additive signed measure such that $\inf\{m(\text{sp}(x)) : x \in \mathcal{S}(H)\} > -\infty$, then m is

bounded and there is a unique Hermitian operator T on H such that (2.1) holds, see [She, Dvu1] or [Dvu3, Thm 3.2.16].

A finitely additive measure m is σ -finite if there is a sequence $\{M_i\}$ of mutually orthogonal subspaces of H such that $\bigvee_i M_i = H$ and every $m(M_i)$ is finite.

A function $f : \mathcal{S}(H) \rightarrow [-\infty, \infty]$ is said to be a *frame function* if, for any $M \in \mathcal{L}(H)$, there is a constant $W_M \in [-\infty, \infty]$ such that $\sum_i f(x_i) = W_M$ holds for any ONB $\{x_i\}$ in M . The constant W_M is said to be the *weight* of f on $M \in \mathcal{L}(H)$. It is worthy of recalling that (i) if $W_H = \infty$, then $W_M > -\infty$ for any $M \in \mathcal{L}(H)$, (ii) if $W_H = -\infty$, then $W_M < \infty$ for any $M \in \mathcal{L}(H)$. In what follows, we will study frame functions f such that $f(x) > -\infty$ for any $x \in \mathcal{S}(H)$.

We note that if $f : \mathcal{S}(H) \rightarrow \mathbb{R}$ has the property that there is a constant $W \in \mathbb{R}$ such that $\sum_i f(x_i) = W$ for any ONB $\{x_i\}$, then f is a frame function. Indeed, if $M \in \mathcal{L}(H)$ is given, take two ONB's $\{x_i\}$ and $\{y_i\}$ of M and let $\{z_j\}$ be an ONB in M^\perp . Then $\{x_i\} \cup \{z_j\}$ and $\{y_i\} \cup \{z_j\}$ are ONBs of H . Hence, $\sum_i f(x_i) + \sum_j f(z_j) = W = \sum_i f(y_i) + \sum_j f(z_j)$ and the series converge absolutely, so that $\sum_i f(x_i) = \sum_i f(y_i)$.

Let m be a completely additive signed measure. Then the function $f(x) := m(\text{sp}(x))$, $x \in \mathcal{S}(H)$, is a frame function. Conversely, any frame function f defines a completely additive measure on $\mathcal{L}(H)$. Indeed, let $\{x_i\}$ be an ONB of $M \in \mathcal{L}(H)$. Then the mapping $m(M) := \sum_i f(x_i) = W_M$, $M \in \mathcal{L}(H)$, where W_M is the weight of f on M is a completely additive measure.

Let S be a linear submanifold of H dense in H . A function $f : \mathcal{S}(S) \rightarrow [-\infty, \infty]$ is said to be a *frame type function* on H if (i) for any ONS $\{x_i\}$ in S , either $\{f(x_i)\}$ is summable or $\sum_i f(x_i) = \infty$, and (ii) for any finite-dimensional subspace K of S , $f|_{\mathcal{S}(K)}$ is a frame function on K . To show the relationship between a frame type function $f : \mathcal{S}(S) \rightarrow [-\infty, \infty]$ and a dense submanifold S , we will write $f \sim (f, S)$. If we change the latter (i) to (i)' for any ONS $\{x_i\}$ in S , either $\{f(x_i)\}$ is summable or $\sum_i f(x_i) = -\infty$, then $-f$ satisfies (i), and vice-versa.

We note that there is a result from theory of series, [Fic, Thm 388, p. 319], Riemann's Rearrangement Theorem, saying that if $\{a_n\}$ is an infinite sequence of real numbers such that $\sum_n a_n$ is finite but $\sum_n |a_n| = \infty$, then for any $a \in [-\infty, \infty]$, there is a rearrangement $\{a_{n'}\}$ of $\{a_n\}$ such that $a = \sum_{n'} a_{n'}$.

Therefore, let f be a frame type function and $\{x_i\}$ be an ONS in S . By the Riemann Rearrangement Theorem, if $\{f(x_i)\}$ is summable, then $\sum_i |f(x_i)| < \infty$, otherwise we can rearrange $\{x_i\}$ to $\{x_{i'}\}$ such that $\sum_{i'} f(x_{i'}) = -\infty$ which is absurd.

It is important to emphasize that frame type functions were originally introduced in [DoSh] (cf. [Dvu3, Sect 3.2.4]) in a stronger form than our notion, because in [DoSh] it was supposed that (i) has the form “ $\{f(x_i)\}$ is summable for any ONS $\{x_i\}$ in S .” We say that a frame type function $f \sim (f, S)$ is *finite* on $\mathcal{S}(S)$ if, for every $x \in \mathcal{S}(S)$, we have $-\infty < f(x) < \infty$.

We note that if $f \sim (f, S)$ is a finite frame type function, $x \in \mathcal{S}(S)$ and $|\lambda| = 1$, then $f(x) = f(\lambda x)$.

It is clear that any frame function on H such that $W_H > -\infty$ is a frame type function on H . In addition, if f is a frame type function on H with given S , then if K is a closed subspace of H such that $S \cap K$ is dense in K , then $f_K := f|_{\mathcal{S}(S \cap K)}$ is a frame type function on K .

Example 2.3. Let H be a complex separable Hilbert space, $\dim H = \aleph_0$.

- (i) If T is a Hermitian trace operator, then $f(x) = (Tx, x)$, $x \in \mathcal{S}(H)$, is a frame function on H .
- (ii) If T_1 is a positive unbounded operator and T is a positive Hermitian trace operator, then $f(x) = (T_1^{1/2}x, T_1^{1/2}x) - (Tx, x)$, $x \in D(T_1^{1/2})$, is a finite frame type function with $S = D(T_1^{1/2})$, where $D(T_1^{1/2})$ is the domain of definition of $T_1^{1/2}$.
- (iii) Let M be a fixed closed subspace of H , $M \subset H$, $M \neq H$, let T_M be a trace operator on M . Then

$$f(x) = \begin{cases} (T_M x, x) & \text{if } x \in M, \\ \infty & \text{otherwise,} \end{cases}$$

is an unbounded frame type function with $S = H$.

- (iv) Let $\{e_i\}$ be an ONB and let us define a diagonal operator T on H such that $Te_i = (-1)^i e_i$. Then T is a Hermitian operator defined everywhere on H and it defines a bounded symmetric bilinear form $f(x) = (Tx, x)$, $x \in \mathcal{S}(H)$, with $S = H$, that is not a frame type function.

The case (iv) can be generalized as follows. Let $\{a_n\}$ be a bounded sequence of real numbers. We set $a_n^+ = \max\{a_n, 0\}$ and $a_n^- = -\inf\{a_n, 0\}$. Let $\{e_n\}$ be a fixed ONB and we define three bounded Hermitian operators T , T^+ and T^- on H by $Te_n = a_n e_n$, $T^+ e_n = a_n^+ e_n$, and $T^- e_n = a_n^- e_n$. Then $T = T^+ + T^-$, and T^+ and T^- are positive and negative parts of T , respectively. Using the Riemann Rearrangement Theorem, we have the following four cases:

- (I) $\sum_n a_n$ is convergent, $\sum_n |a_n| < \infty$.
- (II) $\sum_n a_n$ is convergent, $\sum_n |a_n| = \infty$.
- (III) $\sum_n a_n = \infty$, and $\sum_n a_n^+ = \infty$, $\sum_n a_n^- < \infty$ or $\sum_n a_n^+ < \infty$, $\sum_n a_n^- = \infty$.
- (IV) $\sum_n a_n^+ = \infty = \sum_n a_n^-$.

If we set $f(x) = (Tx, x)$, $x \in \mathcal{S}(H)$, then in case (I), f is a finite bounded frame function, in cases (II) and (IV) it is no frame type function.

Case (III): Let $s_n = a_1 + \dots + a_n$, $n \geq 1$. Then $s_n = s_n^+ - s_n^-$, where $s_n^+ = a_1^+ + \dots + a_n^+$ and $s_n^- = a_1^- + \dots + a_n^-$, $n \geq 1$. Then either $\{s_n^+\}$ is convergent and $\{s_n^-\}$ is divergent or vice-versa. Let $\{x_i\}$ be an arbitrary ONB of H . The first possibility implies $T^+ \in \text{Tr}(H)$ and $T^- \notin \text{Tr}(H)$. Then $-(T^- x_i, x_i) \leq (Tx_i, x_i) \leq (T^+ x_i, x_i)$, and this gives $T^+ \notin \text{Tr}(H)$ which is absurd. Thus we have only the second possibility, i.e., $T^+ \notin \text{Tr}(H)$ and $T^- \in \text{Tr}(H)$.

Let $K > 0$ be given. Since $\sum_i (T^+ x_i, x_i) = \infty$, there is an integer $n_0 > 0$ such that, for every integer $n \geq n_0$, we have $\sum_{i=1}^n (T^+ x_i, x_i) > K + \text{tr}(T^-)$ so that $\sum_{i=1}^n (Tx_i, x_i) = \sum_{i=1}^n (T^+ x_i, x_i) - \sum_{i=1}^n (T^- x_i, x_i) > K + \text{tr}(T) - \sum_{i=1}^n (T^- x_i, x_i) > K$ which yields $\sum_i (Tx_i, x_i) = \infty$.

Now let $\{z_j\}$ be an arbitrary ONS in H . We have two cases (i) $\sum_j (T^+ z_j, z_j) = \infty$ and (ii) $\sum_j (T^+ z_j, z_j) < \infty$. In the case (i), using the same way as that for $\{x_i\}$, we have $\sum_j (Tz_j, z_j) = \infty$, and in the case (ii), we have $\sum_j (Tz_j, z_j)$ is convergent. Consequently, we have proved the following statement:

Proposition 2.4. *Let $\{a_n\}$ be a bounded sequence of real numbers, $\{e_n\}$ be a fixed ONB on a Hilbert space H , $\dim H = \aleph_0$, and let $Te_n = a_n e_n$, $n \geq 1$. Then $f(x) = (Tx, x)$, $x \in \mathcal{S}(H)$, defines a frame type function if and only if either $\sum_n |a_n| < \infty$ or $\sum_n a_n^+ < \infty$, $\sum_n a_n^- = \infty$. The first case implies f is a bounded frame function and the second one implies f is a bounded frame type function.*

It is important to note that if $\dim H > 1$ is finite, there are unbounded frame functions on H , [She], [Dvu3, Prop 3.2.4]. If H is infinite-dimensional, then the surprising result [DoSh] asserts that any finite frame type function f such that $\{f(x_i)\}$ is summable for any ONS $\{x_i\}$ in S is necessarily bounded, see also [Dvu3, Thm 3.2.20].

A frame type function f on H defined on a dense linear submanifold S of H is said to be (i) *semibounded* if $\inf\{f(x) : x \in \mathcal{S}(S)\} > -\infty$, and (ii) *bounded* if $\sup\{|f(x)| : x \in \mathcal{S}(S)\} < \infty$.

Semibounded frame type functions are connected with semibounded bilinear forms as it follows from the following theorem.

Theorem 2.5. *Let $f \sim (f, S)$ be a semibounded finite frame type function on H , $\dim H \neq 2$. There is a unique semibounded symmetric bilinear form t with $D(t) = S$ and $f(x) = t(x, x)$, $x \in \mathcal{S}(S)$.*

Proof. First, let H be finite-dimensional and $\dim H \neq 2$. Then $H = S$ and f is a frame function. Since f is semibounded, then f is bounded on $\mathcal{S}(H)$. Indeed, let y be any unit vector of H and complete it by unit vectors x_1, \dots, x_n to be $\{y, x_1, \dots, x_n\}$ an ONB of H . Let $K = \inf\{f(x) : x \in \mathcal{S}(H)\}$. Then $|f(y)| \leq W_H + n|K|$. By Gleason's Theorem [Dvu3, Thm 3.2.1], there is a unique Hermitian operator T on H such that $f(x) = (Tx, x)$, $x \in \mathcal{S}(H)$, and if we set $t(x, x) = (Tx, x)$, $x \in H$, then t is a bounded symmetric bilinear form such with $D(t) = S = H$ and $f(x) = t(x, x)$, $x \in \mathcal{S}(H)$.

Now let $\dim H = \infty$. Let N be a finite-dimensional subspace of S , $\dim N \geq 3$. As in the first part of the present proof, f is bounded on $\mathcal{S}(N)$. There is a Hermitian operator $T_N : N \rightarrow N$ and a bounded symmetric bilinear form t_N on $N \times N$ such that $f(x) = t_N(x, x) = (T_N x, x)$, $x \in \mathcal{S}(N)$. We define a bilinear form t on $S \times S$ as follows: Let x and y be two vectors of S and let N be any finite-dimensional subspace of S , $\dim N \geq 3$, containing both x and y . Put $t(x, y) = t_N(x, y)$. This t is a well-defined bilinear form on $S \times S$ since if M is another finite-dimensional subspace of S , $\dim M \geq 3$, containing x and y , then for $Q = M \vee N$, we have $t_N(x, y) = t_Q(x, y) = t_M(x, y)$. It is evident that $f(x) = t(x, x)$, $x \in \mathcal{S}(S)$.

Then t is a semibounded symmetric bilinear form on H with $D(t) = S$. The uniqueness of t follows from the Gleason Theorem. \square

We note that if $f \sim (f, S)$ is a frame type function such that $\{f(x_i)\}$ is summable for any ONB $\{x_i\}$ in S and $\dim S \neq 2$, then Theorem 2.5 can be strengthened as follows (cf. [Dvu3, Thm 3.2.21]): There is a unique Hermitian trace operator T on H such that

$$f(x) = (Tx, x), \quad x \in \mathcal{S}(S).$$

The proof of the following important lemma by Lugovaya–Sherstnev can be found in [LuSh], see also [Dvu3, Lem 3.4.2, Cor 3.4.3].

Lemma 2.6 (Lugovaya–Sherstnev). *Let $\dim H = 3$ and let f be a frame function on H . Let there be two orthogonal unit vectors x and y such that $f(x)$ and $f(y)$ are finite. If $W_H = \infty$ and $f(z)$ is finite, then $z \in \text{sp}(x, y)$.*

Theorem 2.7. *Let $\dim H \neq 2$ and f be a semibounded frame function on H such that there is an ONB $\{x_i\}$ with $f(x_i)$ finite for any unit vector x_i . There is a unique semibounded symmetric bilinear form t with $D(t)$ dense in H such that*

$$f(x) = \begin{cases} t(x, x) & \text{if } x \in \mathcal{S}(D(t)), \\ \infty & \text{if } x \in \mathcal{S}(H) \setminus \mathcal{S}(D(t)). \end{cases} \quad (2.2)$$

Proof. If H is finite-dimensional, W_H and f are finite, the proof follows the same ideas as the first part of the proof of Theorem 2.5 with $D(t) = H$.

Now let $\dim H = \infty$ and let $\{x_i\}$ be an ONB of H such that every $f(x_i)$ is finite. Denote by $D(f) := \{x \in H \setminus \{0\} : f(x/\|x\|) \text{ is finite}\} \cup \{0\}$. Then $x_i \in D(f)$ for any x_i . We assert that $D(f)$ is a linear submanifold of H . Indeed, let $x, y \in D(f)$ be two nonzero vectors. Without loss of generality, we can assume x and y are two unit vectors from $D(f)$ that are linearly independent. If $x \perp y$, then clearly $x + y \in \text{sp}(x, y)$. Now let $x \not\perp y$.

Take two vectors x_1, x_2 from the ONB $\{x_i\}$ and set $P = \text{sp}(x_1, x_2)$. If $x, y \in P$, then $x + y \in P$ and $x + y \in D(f)$. Let $x \notin P$ and define $M_1 = \text{sp}(x_1, x_2, x)$. We assert the weight W_{M_1} of f on M_1 is finite; if not, by the Lemma Lugovaya–Sherstnev we conclude $x \in P$ which is absurd.

If $y \in M_1$, then clearly $x + y \in M_1$ and $x + y \in D(t)$. Assume finally $y \notin M_1$. Choose a unit vector $z \in M_1$ that is orthogonal to x . Again by Lemma 2.6, we conclude that f on $M_2 = \text{sp}(x, y, z)$ has finite weight, so that $x + y \in D(f)$.

Since $D(f)$ contains the ONB $\{x_i\}$, $S := D(f)$ is dense in H . Since $f|_S \sim (f|_S, S)$ is a semibounded finite frame type function on H , by Theorem 2.5, there is a unique semibounded symmetric bilinear form t with dense $D(t) = S = D(f)$ such that $f(x) = t(x, x)$, $x \in \mathcal{S}(S)$. Hence, (2.2) holds. \square

It is clear that if t is a positive symmetric bilinear form with $D(t)$ dense in H , then $f(x) := t(x, x)$, $x \in \mathcal{S}(D(t))$, is a finite frame type function on H . Thus for positive finite frame type functions and positive symmetric bilinear forms with dense domain there is a one-to-one correspondence. It is worthy of attention that for non-positive bilinear forms such a correspondence is not true, in general, as it follows from Example 2.3(iv), where there is presented a bounded bilinear form not positive that is not a frame type function.

If t is a positive symmetric bilinear form with dense domain, then f defined on $\mathcal{S}(H)$ by (2.2) is not necessarily a frame function. For example, in [Lug1], see also [Dvu3, Thm 3.7.2], it is shown that if t is a singular positive symmetric bilinear form with dense domain, then f defined by (2.2) is not a frame function, but $f|_{D(t)}$ is a positive finite frame type function.

We note that according to [DoSh], [Dvu3, Thm 3.2.20], every finite frame type function $f \sim (f, S)$, such that $\{f(x_i)\}$ is summable for any ONS $\{x_i\}$ in S , is bounded. We do not know whether every (our) finite frame type function is semibounded if $\dim H = \infty$. As it was already mentioned, according to [Dvu3, Prop 3.2.4], there are unbounded frame type functions for H , $\dim H = n \geq 2$, that are not generated by any bilinear form.

3. GLEASON MEASURES ON $\mathcal{L}(H)$ AND GENERALIZED EFFECT ALGEBRAS

In this section, we study finitely additive measures that are regular. Different classes of measures will be studied from the point of view of generalized effect algebras.

A finitely additive measure m on $\mathcal{L}(H)$ is said to be (i) *regular* if

$$m(M) = \sup\{m(P) : P \subseteq M, P \in \mathcal{L}(H), \dim P < \infty\}, \quad M \in \mathcal{L}(H),$$

and (ii) $\mathcal{P}(H)_1$ -bounded if $\sup\{m(\text{sp}(x)) : x \in D(m)\} < \infty$, where

$$D(m) := \{x \in H : m(\text{sp}(x)) < \infty\} \cup \{\mathbf{0}\}.$$

According to [Dvu3], m satisfies (i) the *L-S property* (L-S stands for Lugovaja and Sherstnev), if there is a two-dimensional subspace Q of H such that $m(Q) < \infty$, (ii) the *density property* if $D(m)$ is dense in H , and (iii) the *L-S density property* if both (i) and (ii) hold.

We note that the L-S property entails that $D(m)$ is a dense linear subspace of H . If o is the zero measure o , i.e. $o(M) = 0$, $M \in \mathcal{L}(H)$, then o is a regular measure with the L-S property. In addition, if $\dim H \neq 2$, then a finitely additive measure m has the L-S density property iff m is σ -finite.

We denote by $\text{Reg}(H)$ the class of regular measures with the L-S property.

An important relationship between regular finitely additive measures with the L-S property and positive bilinear forms with dense domain is the following result, for the proof see [Dvu3, Lem 3.7.8. Thm 3.7.9]. Before that we remind an important notation: Let t be a bilinear form with domain $D(t)$, let $M \in \mathcal{L}(H)$ be given, and let P_M be the orthogonal projector of H onto M . We define a new bilinear form $t \circ P_M$ as a bilinear form whose domain is the set $D(t \circ P_M) := \{x \in H : P_M x \in D(t)\}$. If this bilinear form is determined by a trace operator T_M^t , we write $t \circ P_M \in \text{Tr}(H)$ and $\text{tr}(t \circ P_M) := \text{tr}(T_M^t)$.

Theorem 3.1. *Let H be a Hilbert space.*

(1) *Let t be a positive bilinear form such that $D(t)$ is dense in H . Then the mapping $m_t : \mathcal{L}(H) \rightarrow [0, \infty]$ given by*

$$m_t(M) = \begin{cases} \text{tr}(t \circ P_M) & \text{if } t \circ P_M \in \text{Tr}(H), \\ \infty & \text{otherwise,} \end{cases} \quad (3.1)$$

is a regular finitely additive measure with the L-S density property.

(2) *Let m be a regular finitely additive measure with the L-S density property on $\mathcal{L}(H)$, $\dim H \neq 2$. Then there exists a unique bilinear form t with domain $D(t) = D(m)$ such that (3.1) holds.*

Since formula (3.1) is a generalization of the famous Gleason formula (2.1), we call measures expressed by bilinear forms via (3.1) also Gleason measures.

We note that by Theorem 3.1, if m is a regular finitely additive measure with the L-S property, there is a unique positive bilinear form t with $D(t) = D(m)$ such that (3.1) holds. In other words, if $x \in D(m)$, we have $t(x, x) = \|x\|^2 m(\text{sp}(x))$ if $x \neq \mathbf{0}$, otherwise, $t(\mathbf{0}, \mathbf{0}) = 0$.

On the other hand, if m is a finite positive regular measure on $\mathcal{L}(H)$, then m is completely additive [Dvu3, Thm 3.7.2]. For unbounded measures this is not a case, in general. More precisely, any positive singular bilinear form t generates by (3.1) a regular measure that is not completely-additive, see [Dvu3, Thm 3.7.6], [Lug2]. A criterion when a bilinear form t generates a completely additive measure on $\mathcal{L}(H)$, see [Lug2], [Dvu3, Thm 3.7.5]: A positive bilinear form t defines through (3.1) a σ -finite completely additive measure on $\mathcal{L}(H)$ iff for any $M \in \mathcal{L}(H)$,

$$\overline{(t \circ P_M)_r} \in \text{Tr}(H) \text{ implies } t \circ P_M \in \text{Tr}(H), \quad (3.2)$$

where $\overline{(t \circ P_M)_r}$ is the closure of the regular part of $t \circ P_M$.

Proposition 3.2. *If m_1, m_2 are regular measures on $\mathcal{L}(H)$, then $m_1 + m_2$ is also a regular measure.*

Proof. Given $M \in \mathcal{L}(H)$ and $i = 1, 2$, there are two sequences $\{P_n^1\}$ and $\{P_n^2\}$ of finite-dimensional subspaces of M such that $m_i(M) = \lim_n m_i(P_n^i)$. Then $m_i(M) = \lim_n m_i(P_n^1 \vee P_n^2)$ which entails $(m_1 + m_2)(M) = m_1(M) + m_2(M) = \lim_n m_1(P_n^1 \vee P_n^2) + \lim_n m_2(P_n^1 \vee P_n^2) = \lim_n (m_1 + m_2)(P_n^1 \vee P_n^2)$ which proves regularity of $m_1 + m_2$. \square

Proposition 3.2 suggests to study the set of regular Gleason measures also from the point of view of generalized effect algebras.

A partial algebra $(E; \oplus, 0)$ is called a *generalized effect algebra* if $0 \in E$ is a distinguished element and \oplus is a partially defined binary operation on E which satisfies the following conditions for any $x, y, z \in E$:

- (GEi) $x \oplus y = y \oplus x$, if one side is defined,
- (GEii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$, if one side is defined,
- (GEiii) $x \oplus 0 = x$,
- (GEiv) $x \oplus y = x \oplus z$ implies $y = z$ (cancellation law),
- (GEv) $x \oplus y = 0$ implies $x = y = 0$.

For every generalized effect algebra E , a partial binary operation \ominus and a relation $\leq := \leq_\oplus$ can be defined by

- (ED) $x \leq y$ and $y \ominus x = z$ iff $x \oplus z$ is defined and $x \oplus z = y$.

Then \leq is a partial order on E under which 0 is the least element of E . A generalized effect algebra with the top element $1 \in E$ is called an *effect algebra* and we usually write $(E; \oplus, 0, 1)$ for it.

For example, let G be an Abelian po-group, i.e. a group $G = (G; +, 0)$ endowed with a partial order \leq such that $a \leq b$, then $a + c \leq b + c$ for each $c \in G$. Let $G^+ := \{g \in G : g \geq 0\}$ be the positive cone of G . Then $(G^+; \oplus, 0)$ is a generalized effect algebra, where $a \oplus b = a + b$, $a, b \in G^+$. If $u \in G^+$ is a fixed element, then the interval $[0, u] := \{g \in G : 0 \leq g \leq u\}$ defines an effect algebra $([0, u]; \oplus, 0, u)$, where $a \oplus b = a + b$, $a, b \in [0, u]$, whenever $a + b \leq u$.

In particular, if $\mathcal{B}(H)$ is the class of all Hermitian operators of a Hilbert space H , then $\mathcal{B}(H)$ is a po-group, saying $A \leq B$ iff $(Ax, x) \leq (Bx, x)$, $x \in H$, and $\mathcal{B}(H)^+$ is a generalized effect algebra. If $\mathcal{E}(H) := \{A \in \mathcal{B}(H) : 0 \leq A \leq I\}$, where 0 and I are the zero and identity operators, respectively, then $\mathcal{E}(H)$ is a prototypical example of effect algebras important for Hilbert space quantum mechanics.

A subset S of E is called a *sub-generalized effect algebra* (*sub-effect algebra*) of E iff (i) $0 \in S$ ($1 \in S$), (ii) if out of elements $x, y, z \in E$ such that $x \oplus y = z$ at least two are in S , then all $x, y, z \in S$.

The following theorem follows ideas from [RZP], where a structure of linear operators was described.

Theorem 3.3. *Let H be an infinite-dimensional complex Hilbert space. Let $\text{Reg}_f(H)$ be the set of regular finitely additive measures m on $\mathcal{L}(H)$ with the L-S density property such that if m is $\mathcal{P}_1(H)$ -bounded, then $D(m) = H$. Let us define a partial operation \oplus on $\text{Reg}_f(H)$: For $m_1, m_2 \in \text{Reg}_f(H)$, $m_1 \oplus m_2$ is defined if and only if m_1 or m_2 is $\mathcal{P}_1(H)$ -bounded or $D(m_1) = D(m_2)$ and then $m_1 \oplus m_2 := m_1 + m_2$. Then $(\text{Reg}_f(H); \oplus, 0)$ is a generalized effect algebra.*

Proof. Let $m_1, m_2 \in \text{Reg}_f(H)$ be measures such that $m_1 \oplus m_2$ is defined. By Theorem 3.1, there are unique positive bilinear forms t_1, t_2 with $D(t_1) = D(m_1), D(t_2) = D(m_2)$ such that (3.1) holds. Using [DvJa, Thm 4.1], we have that $t_1 + t_2$ is a densely defined positive bilinear form on $D(t_1 + t_2) = D(t_1) \cap D(t_2)$. Let $M \in \mathcal{L}(H)$ be such that $m_1(M), m_2(M) < \infty$. Then $(m_1 \oplus m_2)(M) = m_1(M) + m_2(M) = \text{tr}(t_1 \circ P_M) + \text{tr}(t_2 \circ P_M) = \text{tr}(t_1 \circ P_M + t_2 \circ P_M) = \text{tr}((t_1 + t_2) \circ P_M)$. If $m_1(M) = \infty$ or $m_2(M) = \infty$, we have $t_1 \circ P_M \notin \text{Tr}(H)$ or $t_2 \circ P_M \notin \text{Tr}(H)$ which yields $(t_1 + t_2) \circ P_M \notin \text{Tr}(H)$. Therefore, $m_1 \oplus m_2$ is determined by $t_1 + t_2$ in the sense of (3.1) i.e. it is regular and possesses the L-S density property.

(GEi) It follows from the commutativity of the usual sum $+$.

(GEii) Let $m_1, m_2, m_3 \in \text{Reg}_f(H)$ such that $(m_1 \oplus m_2) \oplus m_3$ is defined. First, let us assume that $D(m_1) = H$. Then $D(m_1) \cap D(m_2) = D(m_2)$ and $D(m_2) = D(m_3)$ or $D(m_3) = H$. That is, $m_2 \oplus m_3$ is defined with $D(m_2 + m_3) = D(m_2)$, hence also $m_1 \oplus (m_2 \oplus m_3)$ is defined. The same analogy holds for $D(m_2) = H$. If $D(m_1), D(m_2) \neq H$, we have $D(m_1) = D(m_2)$ and then $D(m_3) = D(m_2)$ or $D(m_3) = H$, which yields the existence of $m_1 \oplus (m_2 \oplus m_3)$. The equality follows from the associativity of the usual sum $+$.

(GEiii) It holds since $o \in \text{Reg}_f(H)$ is $\mathcal{P}_1(H)$ -bounded, hence $D(o) = H$.

(GEiv) Let $m_1 \oplus m_2 = m_1 \oplus m_3$. By Theorem 3.1, for any $i = 1, 2, 3$, there is a unique positive bilinear form t_i with $D(t_i) = D(m_i)$ such that (3.1) holds.

Let m_1, m_2, m_3 be all $\mathcal{P}_1(H)$ -bounded. Then $t_1(x, y) + t_2(x, y) = t_1(x, y) + t_3(x, y)$ for all $x, y \in H$, which is equivalent to $t_2(x, y) = t_3(x, y)$ for all $x, y \in H$ hence $t_2 = t_3$, so that $m_2 = m_3$ by Theorem 3.1.

Let m_1 be $\mathcal{P}_1(H)$ -bounded and let m_2 be not $\mathcal{P}_1(H)$ -bounded. Then m_3 has to be not $\mathcal{P}_1(H)$ -bounded and $D(m_2) = D(m_1 + m_2) = D(m_1 + m_3) = D(m_3)$. By the same argument as in the previous case (restricted on $D(m_2)$) we have $t_2 = t_3$ and consequently, $m_2 = m_3$.

Let m_1 be not $\mathcal{P}_1(H)$ -bounded and let m_2 be $\mathcal{P}_1(H)$ -bounded. Then $D(m_1) = D(m_1 + m_2) = D(m_1 + m_3)$. For every $x, y \in D(t_1)$, we have $t_1(x, y) + t_2(x, y) = t_1(x, y) + t_3(x, y)$. Hence, $t_2(x, y) = t_3(x, y)$, that is $t_3|_{D(t_1)} = t_2|_{D(t_1)}$. Since t_2 is on $D(t_1)$ bounded, t_3 is also bounded and by [DvJa, Lem 4.1], t_3 can be extended on H in a unique way to a symmetric bilinear form, that is $t_2 = t_3$ which implies $m_2 = m_3$.

Let m_1 and m_2 be not $\mathcal{P}_1(H)$ -bounded. Then $D(m_1) = D(m_2) = D(m_1 + m_2) = D(m_1 + m_3)$ and in the same way as in the previous case, $t_3|_{D(t_1)} = t_2|_{D(t_1)}$. Since m_2 is not $\mathcal{P}_1(H)$ -bounded, m_3 is also not $\mathcal{P}_1(H)$ -bounded. Because $m_1 \oplus m_3$ is defined, we have $D(t_1) = D(t_3) = D(t_2)$, hence $t_3 = t_2$ and consequently, $m_3 = m_2$.

(GEv) Assume that there is $m_1, m_2 \in \text{Reg}_f(H)$ satisfying $m_1 \oplus m_2 = o$. Then $D(m_1 \oplus m_2) = D(o) = H$ and there exist unique positive bilinear forms t_1, t_2 with $D(t_1) = D(t_2) = H$ such that (3.1) holds. Using [DvJa, Thm 4.1], we have $t_1 = t_2 = o$ which means $m_1 = m_2 = o$. \square

Theorem 3.4. *Let H be a separable infinite-dimensional complex Hilbert space. Let $\text{Reg}_f^\sigma(H)$ be the set of σ -additive measures from $\text{Reg}_f(H)$. Then $(\text{Reg}_f^\sigma(H); \oplus_{|\text{Reg}_f^\sigma(H)}, o)$ is a generalized effect algebra, but $\text{Reg}_f^\sigma(H)$ is not a sub-generalized effect algebra of the generalized effect algebra $(\text{Reg}_f(H); \oplus, o)$.*

Proof. Clearly, $o \in \text{Reg}_f^\sigma(H)$. It is straightforward to show that if a sum $m_1 \oplus m_2$ of two measures from $\text{Reg}_f^\sigma(H)$ exists in $\text{Reg}_f(H)$, it is a measure from $\text{Reg}_f^\sigma(H)$ as

well. Let us consider a regular measure $m_1 \in \text{Reg}_f(H)$ given by $m_1(M) = \dim M$, $M \in \mathcal{L}(H)$, which is a $P_1(H)$ -bounded σ -additive measure and it corresponds to the bilinear form t_I corresponding to the identity operator I on H , i.e. $t_I(x, x) = (Ix, x) = (x, x)$, $x \in H$. Let $m_2 \in \text{Reg}_f(H)$ be a finitely additive measure given by (3.1) for some positive singular bilinear form s with $D(s) = H$, hence by [Dvu3, Thm 3.7.2], m_2 is not σ -additive. Since $D(s) = H$, it holds $s \circ P_M \in \text{Tr}(H)$ if $\dim M < \infty$. Moreover, $t_I \circ P_M = P_M \in \text{Tr}(H)$ iff $\dim M < \infty$ which gives $(s + t_I) \circ P_M \in \text{Tr}(H)$ iff $\dim M < \infty$. By [Sim, Cor 2.3], we have $(s + t_I)_r = t_I$ and using (3.2) or [Dvu3, Thm 3.7.5], $m_1 \oplus_{|\text{Reg}_f^\sigma} m_2 := m_1 \oplus m_2 = m_1 + m_2$ is a σ -additive measure. Which means that $\text{Reg}_f^\sigma(H)$ is not a sub-generalized effect algebra of $\text{Reg}_f(H)$ because $m_1, m_1 \oplus m_2 \in \text{Reg}_f^\sigma(H)$ but $m_2 \notin \text{Reg}_f^\sigma(H)$. \square

Remark 3.5. Let H be an infinite-dimensional complex Hilbert space. Let $\text{Reg}_f^c(H)$ be the set of completely additive measures from $\text{Reg}_f(H)$. Then by [Dvu3, Thm 3.6.3] $\text{Reg}_f^c(H) = \text{Reg}_f(H)$, which yields $(\text{Reg}_f^c(H); \oplus, o)$ is a generalized effect algebra.

Using Theorem 2.5 and Theorem 2.7, we can extend the previous results also for frame type functions. For any two frame type function f_1, f_2 on H , their *usual sum* $f(x) := f_1(x) + f_2(x)$ for all $x \in \mathcal{S}(S_1) \cap \mathcal{S}(S_2)$ is again a frame type function, whenever $S_1 \cap S_2$ is dense in H .

A frame function o_f is defined by $o_f(x) = 0$ for all $x \in \mathcal{S}(H)$.

Theorem 3.6. Let H be an infinite-dimensional complex Hilbert space. Let $\text{Ftf}(H)$ be the set of positive finite frame type functions $f \sim (f, S)$ on H such that whenever f is bounded, then $S = H$. Let us define a partial operation $\hat{\oplus}$ on $\text{Ftf}(H)$: For $f_1, f_2 \in \text{Ftf}(H)$, $f_1 \hat{\oplus} f_2$ is defined if and only if $f_1 \sim (f, S_1)$ or $f_2 \sim (f, S_2)$ is bounded or $S_1 = S_2$ and then $f_1 \hat{\oplus} f_2 := f_1 + f_2$. Then $(\text{Ftf}(H); \hat{\oplus}, o_f)$ is a generalized effect algebra.

Proof. By Theorem 2.5, there is a one-to-one correspondence between positive finite frame type functions and positive bilinear forms given by $f(x) = t(x, x)$ for $x \in D(t)$ and $S = D(t)$. A positive frame type function f is bounded iff the corresponding bilinear form t is bounded iff the corresponding finitely additive measure m is bounded, and clearly $S = D(t) = D(m)$. Hence, the sets $\text{Ftf}(H)$ and $\text{Reg}_f(H)$ from Theorem 3.4 are in a one-to-one correspondence.

For any two frame type functions $f_1, f_2 \in \text{Ftf}(H)$, $f_1 \hat{\oplus} f_2$ is defined if and only if for finitely additive measures m_1, m_2 , given by bilinear forms $t_1(x, x) := f_1(x)$, $t_2(x, x) := f_2(x)$, $m_1 \oplus m_2$ is defined and then $(f_1 \hat{\oplus} f_2)(x) = (m_1 \oplus m_2)(\text{sp}(x))$. That is, the proof follows the same arguments as the proof of Theorem 3.3. \square

Remark 3.7. Let $f \sim (f, S)$ be a frame type function on H . If the function \overline{f} given by $\overline{f}(x) := f(x)$ for all $x \in S$ and $\overline{f}(x) := \infty$ otherwise is a frame function on H , then we say that f induces a frame function \overline{f} .

Theorem 3.8. Let H be an infinite-dimensional complex Hilbert space. Let $\text{Ff}(H) \subseteq \text{Ftf}(H)$ be the set of all frame type functions f on H which induce a frame function \overline{f} . Then $\text{Ff}(H)$ is a subset of $\text{Ftf}(H)$ but it is not a sub-generalized effect algebra of $(\text{Ftf}(H); \hat{\oplus}, o_f)$, but $(\text{Ff}(H); \hat{\oplus}_{|\text{Ff}(H)}, o_f)$ is a generalized effect algebra on its own.

Proof. A restriction of any finite frame function \bar{f} on $S = \{x \in H \mid f(x) < \infty\}$ is a finite frame type function $f \sim (f, S)$, and on the other hand, f induces \bar{f} . Since there is a one-to-one correspondence between the set of positive frame functions $\text{Ff}(H)$ and the set of completely additive measures $\text{Reg}_f^c(H)$, the theorem follows from Theorem 3.4 and Remark 3.5. \square

4. MONOTONE CONVERGENCE PROPERTIES OF σ -ADDITIVE MEASURES

In the section, we will study some monotone Dedekind upwards (downwards) σ -complete properties of generalized effect algebras of regular measures.

In [Dvu3, Thm 3.10.1], there was proved an analogue of the Nikodým convergence theorem which says following:

Theorem 4.1. *Let $\{m_n\}$ be a sequence of finite signed σ -additive measures on $\mathcal{L}(H)$ of a Hilbert space H . If there is a finite limit $m(M) = \lim_{n \rightarrow \infty} m_n(M)$ for any $M \in \mathcal{L}(H)$, then m is a finite signed measure on $\mathcal{L}(H)$, and $\{m_n\}$ is uniformly σ -additive with respect to n .*

We say that a generalized effect algebra E is (i) *monotone Dedekind upwards σ -complete* if, for any sequence $x_1 \leq x_2 \leq \dots$, which is dominated by some element x_0 , i.e. each $x_n \leq x_0$, the element $x = \bigvee_n x_n$ is defined in E (we write $\{x_n\} \nearrow x$), (ii) *monotone Dedekind downwards σ -complete* if, for any sequence $x_1 \geq x_2 \geq \dots$, the element $x = \bigwedge_n x_n$ is defined in E (we write $\{x_n\} \searrow x$). If E is an effect algebra, both later notions are equivalent. In addition, we say that a generalized effect algebra E is *upwards directed* if, given $a_1, a_2 \in E$, there is $a \in E$ such that $a_1, a_2 \leq a$.

Theorem 4.2. *Let H be an infinite-dimensional complex Hilbert space. Then the generalized effect algebra $(\text{Reg}_f(H); \oplus, o)$ is monotone downwards σ -complete, but it is not Dedekind monotone upwards σ -complete.*

Proof. This holds since $(\text{Reg}_f(H); \oplus, o)$ is isomorphic to the generalized effect algebra of all positive bilinear forms on H , which by [DvJa, Thm 5.2] has the mentioned properties. \square

Theorem 4.3. *Let H be an infinite-dimensional complex Hilbert space. The generalized effect algebra $(\text{Reg}_f^\sigma(H); \oplus, o)$ is neither monotone downwards σ -complete, nor Dedekind monotone upwards σ -complete.*

Proof. Let us consider the complex Hilbert space $H = L^2(0, 1)$ and the following sequence of bilinear forms $\{s_n\}_{n \in \mathbb{N}}$ given by

$$s_n(u, u) := \left(1 + \frac{1}{n}\right) \int_0^1 |u'(x)|^2 dx + |u(0)|^2 + |u(1)|^2, \quad (4.1)$$

where u' is derivative of $u \in L^2[0, 1]$. Let

$$s(u, u) := \int_0^1 |u'(x)|^2 dx + |u(0)|^2 + |u(1)|^2,$$

$$\hat{s}(u, u) := \int_0^1 |u'(x)|^2 dx,$$

$$s_0(u, u) := |u(0)|^2 + |u(1)|^2.$$

Then s, \hat{s} and s_n are closed bilinear forms for all $n \in \mathbb{N}$. According to [Dvu1, Prop 3.7.4], measures $m_s, m_{\hat{s}}$ and m_{s_n} induced by these bilinear forms via (3.1) are σ -additive (due to their regularity also completely additive). Also $s - s_n, \hat{s} - s_n$ are closed bilinear forms, which means that $m_s, m_{\hat{s}} \leq_\sigma m_{s_n}$ for all $n \in \mathbb{N}$, where \leq_σ is the partial order in the generalized effect algebra $\text{Reg}_f^\sigma(H)$ induced by the partial addition in it via (ED). It holds $m_{\hat{s}} \leq_\sigma m_s$ in the generalized effect algebra $\text{Reg}_f(H)$, but since $s - \hat{s}$ is a singular bilinear form, the corresponding measure $m_{s-\hat{s}}$ is not σ -additive hence $m_{\hat{s}} \not\leq_\sigma m_s$. Since $m_s = \bigvee_{n \in \mathbb{N}} m_{s_n}$ in the generalized effect algebra $\text{Reg}_f(H)$, the sequence $\{m_{s_n}\}_{n \in \mathbb{N}}$ has no infimum in $\text{Reg}_f^\sigma(H)$. By [DvJa, Lemma 5.1], $\text{Reg}_f^\sigma(H)$ is not a Dedekind upwards σ -complete generalized effect algebra. \square

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